Eigenspectrum and localization for diffusion with traps

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We investigate the *survival-return* probability distribution and the eigenspectrum for the transition probability matrix, for diffusion in the presence of perfectly absorbing traps distributed with critical disorder in two and three dimensions. The density of states is found to have a Lifshitz tail in the low frequency limit, consistent with a recent investigation of the long-time behavior of the *survival* probability. The localization properties of the eigenstates are found to be very different from diffusion with no traps.

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I. INTRODUCTION

Diffusion in the presence of randomly distributed traps appears in many physical situations [1,2]. Well-known examples include the migration of optical and magnetic excitations in solids [3]. Diffusion with traps differs from the usual *kinetic* random walk in an inhomogeneous environment since each diffusive trajectory must be weighted the same, unlike in the latter case [4–7]. This model, known also as the *ideal chain*, has received much less attention than the kinetic random walk.

For example, although the case of uncorrelated or weakly correlated disorder has been fairly well studied [8–10], only recently was the case of strongly correlated disorder addressed [5–7]. Unlike in the former case, no rigorous results are known for many key quantities in the case of the strongly correlated disorder. For example, the *survival* probability, which is the probability for a walk not to be absorbed by a trap after t steps, is not known rigorously. A recent investigation [6], however, showed that it is possible to extract the asymptotic behavior of the survival probability for the strongly correlated disorder by considering the full probability distribution for the number of surviving walks at the (discrete) time t. The result was found to be qualitatively similar to the case of uncorrelated disorder, the so-called Donsker-Varadhan behavior [10], but quantitatively different with different numerical values of the exponents. Specifically, an asymptotic, stretched-exponential decay of the probability was predicted where the exponent was related to the free energy fluctuation exponent χ.

In this work, we first apply the same method as in [6] to a closely related but slightly different quantity (called the *survival-return* probability, see below), which results in a similar prediction of a stretched-exponential tail. This quantity is, however, directly related to the spectral density of the eigenvalues of the transition probability matrix which describes the diffusion process. This allows us to test the prediction by numerically evaluating the spectral density and comparing with the prediction. In addition, the behavior of the density of states is itself of physical interest since it is in principle experimentally measurable, e.g., by Raman scattering [11].

As a prototype of a system with strongly correlated substitutional disorder, the incipient infinite percolation cluster [12] in two (square lattice) and three dimensions (simple cubic lattice) will be considered. In this model, perfectly absorbing traps are distributed with probability $1-p_c$, where p_c is the critical threshold for the probability for a site to be present in the lattice. The vacancies thus play the role of the traps.

The work is organized as follows. In Section II the analogy to the Schrödinger equation is carried out. In Sec. III the survival-return probability is studied. Section IV establishes the connection with the density of states which is then studied numerically in Sec. V. In Sec. VI the localization properties of the eigenmodes are briefly studied and finally Sec. VII summarizes the conclusions.

II. ANALOGY TO THE SCHRÖDINGER EQUATION

When the total probability is a conserved function of time, the probability $P_{\mathbf{x}_0,\mathbf{x}}(t)$ that a random walker is at the position $\mathbf{x}$ at the (discrete) time t in a d-dimensional hypercubic lattice satisfies the general master equation

$$P_{\mathbf{x}_0,\mathbf{x}}(t+1) = P_{\mathbf{x}_0,\mathbf{x}}(t) + \sum_{\mathbf{y}(\mathbf{x})} [w_{\mathbf{x},\mathbf{y}} P_{\mathbf{x}_0,\mathbf{y}}(t) - w_{\mathbf{y},\mathbf{x}} P_{\mathbf{x}_0,\mathbf{x}}(t)] \,, \qquad (2.1)$$

where $\mathbf{x} \in \mathbb{Z}^d$ is spanned by an orthonormal basis $\{\hat{\mathbf{e}}_\mu\}_{\mu=1,\ldots,d}$ and $w_{\mathbf{x},\mathbf{y}}$ is the transition probability per unit time from $\mathbf{y}$ to $\mathbf{x}$. Site $\mathbf{y}(\mathbf{x})$ denotes a nearest neighbor of $\mathbf{x}$ and the summation is over all nearest neighbors of $\mathbf{x}$. For the present problem with traps, however, we

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assume that the probability at a trap site is immediately lost and that the probability at the next time step is determined solely by the incoming probability from the available neighbor sites. Furthermore, we take a uniform transition probability, i.e., $w_{\mathbf{x},\mathbf{y}} = 1/z$ for available nearest neighbors $\mathbf{x}$ and $\mathbf{y}$ where z is the coordination number of the full lattice. Therefore, denoting the set of available sites by $\mathcal{C}$ and that of traps by $\mathcal{C}^{\perp}$, we get

$$P_{\mathbf{x}_0,\mathbf{x}}(t+1) = \frac{1}{z}\sum_{\mathbf{y}(\mathbf{x})\in\mathcal{C}} P_{\mathbf{x}_0,\mathbf{y}}(t) \quad \text{for } \mathbf{x}\in\mathcal{C} , \tag{2.2}$$

$$P_{\mathbf{x}_0,\mathbf{x}}(t) = 0 \quad \text{for } \mathbf{x}\in\mathcal{C}^{\perp} . \tag{2.3}$$

Using the discrete time derivative defined as

$$\frac{\partial}{\partial t}P_{\mathbf{x}_0,\mathbf{x}}(t) \equiv P_{\mathbf{x}_0,\mathbf{x}}(t+1) - P_{\mathbf{x}_0,\mathbf{x}}(t), \tag{2.4}$$

we can write Eq. (2.2) as

$$\frac{\partial}{\partial t}P_{\mathbf{x}_0,\mathbf{x}}(t) = \frac{1}{z}\sum_{\mathbf{y}}[\delta_{1,|\mathbf{x}-\mathbf{y}|} - z\delta_{\mathbf{y},\mathbf{x}}]P_{\mathbf{x}_0,\mathbf{y}}(t) - \frac{1}{z}V_{\mathbf{x}}P_{\mathbf{x}_0,\mathbf{x}}(t), \tag{2.5}$$

where we defined a hard core potential

$$V_{\mathbf{x}} = \begin{cases} +\infty, & \mathbf{x}\in\mathcal{C}^{\perp} \quad \text{(trap)} , \\ 0, & \mathbf{x}\in\mathcal{C} \quad \text{(available site)} . \end{cases} \tag{2.6}$$

Then we can write Eq. (2.5) as

$$\frac{\partial}{\partial t}P_{\mathbf{x}_0,\mathbf{x}}(t) = -HP_{\mathbf{x}_0,\mathbf{x}}(t) , \tag{2.7}$$

where we defined the Hamiltonian

$$H = \frac{1}{z}(-\nabla^2_{\mathbf{x}} + V_{\mathbf{x}}) . \tag{2.8}$$

The discrete spatial derivative is defined as usual as

$$\nabla_{\mu}P_{\mathbf{x}_0,\mathbf{x}}(t) = P_{\mathbf{x}_0,\mathbf{x}+\hat{\mathbf{e}}_{\mu}}(t) - P_{\mathbf{x}_0,\mathbf{x}}(t). \tag{2.9}$$

By analogy to the case of the Schrödinger equation for a binary random alloy under the tight binding approximation [13], one then expects a stretched-exponential tail (called the Lifshitz tail) for the low-energy behavior of the density of states. The crucial feature for this behavior is the fact that our binary potential represents a random, infinite barrier which isolates the low-energy band and allows the Lifshitz argument to apply. This corresponds to the fact that the traps do not allow a stationary state (frequency $\omega = 0$). In contrast, the case of the kinetic random walk with conserved probability cannot be cast in the form of Eq. (2.7); e.g., for the myopic ant, there is a dispersion in the first term inside the brackets on the right-hand side of Eq. (2.5) and for the blind ant there is one for the second term in the brackets. This corresponds to the fact that the conservation of probability allows a stationary state to exist as well as other states leading up to it. We believe that this is why the density of states in the kinetic random walk problem does not have a Lifshitz tail [14]; rather, it has a power law behavior in the low-energy limit governed by the *spectral dimension* [15].

III. SURVIVAL-RETURN PROBABILITY

The master equation (2.2) can be put in a transfer matrix form and numerically solved by the iteration of this matrix [5,6]. All the quantities of interest can then be computed rather accurately, with the exception of the power moments of multiplicative random variables such as $C \equiv C(\mathbf{x}_0, t)$, which is the number of t-step walks having the common starting point $\mathbf{x}_0$. This is due to the fact that these moments are not self-averaging, which forces us to compute the full probability distribution $P(C,t)$ [6].

A similar feature of the lack of self-averaging holds true for the quantity $C_0 \equiv C_{\mathbf{x}_0,\mathbf{x}_0}(t)$ which is the number of walks that return to the starting point $\mathbf{x}_0$ after t steps. This quantity determines the *survival-return* probability $P_S^0(t)$ by

$$P_S^0(t) = C_{\mathbf{x}_0,\mathbf{x}_0}(t)/z^t. \tag{3.1}$$

In Fig. 1, our numerical result for the distribution of $\ln C_{\mathbf{x}_0,\mathbf{x}}(t)$, denoted by $P(\ln C_0) = P(C_0,t)C_{\mathbf{x}_0,\mathbf{x}_0}(t)$, is shown for the square lattice in $d=2$ for various t, while in Fig. 2 the analogous result for the simple cubic lattice in $d=3$ is shown. In the numerical work, each disorder configuration was generated relative to a seed site, which also served as the starting point $\mathbf{x}_0$ of the chains of 1600 steps to be exactly enumerated. The final results were obtained by averaging over a large number (typically 6000) of independent disorder configurations.

The distribution can be approximated very accurately by a Gaussian; that is, we may express $P(C_0,t)$ by a log-normal distribution of the form

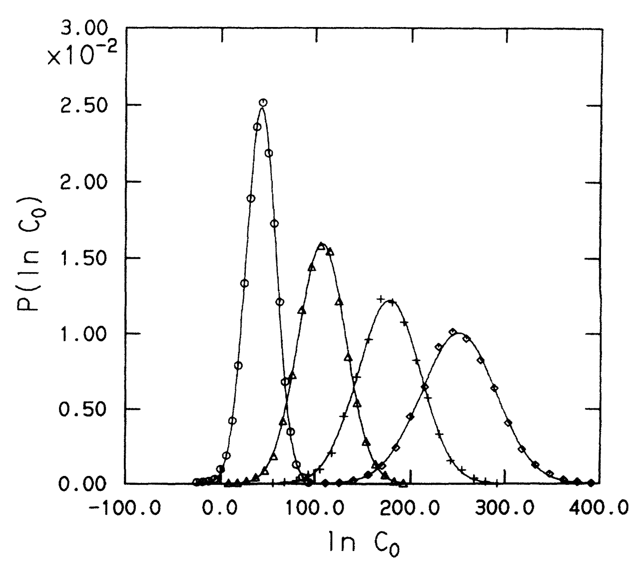

FIG. 1. Calculated distribution $P(\ln C_0)$, for $t=400$ (○), $t=800$ (△), $t=1200$ (+), and $t=1600$ (◇), for the square lattice in $d=2$. The solid curves are the best fit results derived from Eq. (3.2). The values of C_0 were normalized by an arbitrary factor 2.8 for convenience.

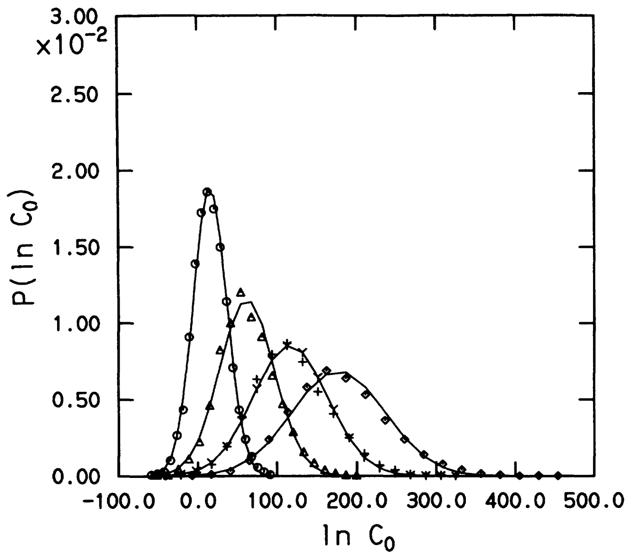

FIG. 2. Calculated distribution $P(\ln C_0)$, for $t = 400$ (○), $t = 800$ (△), $t = 1200$ (+), and $t = 1600$ (◊), for the simple cubic lattice in $d = 3$. The solid curves are the best fit results derived from Eq. (3.2). The values of C_0 were normalized by an arbitrary factor 3.0 for convenience.

$$P(C_0,t) = \frac{1}{C_0\sqrt{2\pi\sigma_t^2}} \exp\left[\frac{-(\ln C_0 - \lambda_t)^2}{2\sigma_t^2}\right] , \tag{3.2}$$

where

$$\lambda_t = \overline{\ln C_{\mathbf{x}_0,\mathbf{x}_0}(t)} , \tag{3.3a}$$

$$\sigma_t^2 = \overline{[\ln C_{\mathbf{x}_0,\mathbf{x}_0}(t)]^2} - \overline{\ln C_{\mathbf{x}_0,\mathbf{x}_0}(t)}^2 . \tag{3.3b}$$

The mean λ_t and the variance σ_t^2 both depend on the time t. This dependence can be deduced from the log-normal fitted but can also be directly evaluated from the logarithmic moments of C_0 which are much easier to obtain numerically than the power moments. The long-time behavior for these two quantities can be fitted rather well by

$$\lambda_t \sim t \ln z_{eff} - \alpha t^{\psi_0} , \tag{3.4a}$$

$$\sigma_t^2 \sim \beta t^{2\chi_0} , \tag{3.4b}$$

where α, β, and z_{eff} are constants.

This behavior is similar to the case of the *survival* probability

$$P_S(t) = C(\mathbf{x}_0, t)/z^t \tag{3.5}$$

as discussed in Ref. [6].

The best fit values of the exponents are summarized in Table I. The calculations based on the log-normal fit and those directly from the logarithmic moments give consistent results as shown. In particular, the value of the *effective* coordination number z_{eff} in $d = 2$ is $z_{eff} = 3.62 \pm 0.01$ (from log-normal fit) and $z_{eff} = 3.60 \pm 0.01$ (from direct calculation). In $d = 3$, we find $z_{eff} = 4.33 \pm 0.14$ and $z_{eff} = 4.12 \pm 0.11$ from the two procedures, respectively. The higher relative value z_{eff}/z in $d = 2$ than $d = 3$ is not surprising and is due to the much more compact structure of the incipient infinite cluster in two dimensions. The effect of the traps is indeed to force the diffusion into regions of high connectivity in order to maximize the entropy [6,7].

TABLE I. Summary of the exponents ψ_0 and χ_0 defined in the text. The labels (D) and (I) mean direct evaluation and from the log-normal distribution, respectively.

d	$\psi_0(D)$	$\psi_0(I)$	$\chi_0(D)$	$\chi_0(I)$
2	0.70 ± 0.01	0.69 ± 0.01	0.65 ± 0.01	0.65 ± 0.01
3	0.78 ± 0.03	0.83 ± 0.03	0.72 ± 0.01	0.70 ± 0.01

The fact that $\sigma_t^2 \gg \lambda_t$ for sufficiently long times has a consequence that the log-normal distribution has to be cut off both at the short and long times as discussed in [6] for the case of the survival probability. If this were not the case, all the moments of $C_{\mathbf{x}_0,\mathbf{x}_0}(t)$, which can be easily computed from Eq. (3.2), would grow faster than z^t, which is clearly impossible. Therefore, we assume a cutoff $1 \le C_{\mathbf{x}_0,\mathbf{x}_0}(t) \le z^t$ (which is certainly true), and carry out the calculation of the moments in the same manner as shown in Ref. [6,7] for $C(\mathbf{x}_0, t)$. This calculation yields a Donsker-Varadhan type behavior [10] for the first moment of $C_{\mathbf{x}_0,\mathbf{x}_0}(t)/z^t$,

$$\ln \overline{P_S^0(t)} \sim -t^{2(1-\chi_0)} , \tag{3.6}$$

where χ_0 is the exponent appearing in Eq. (3.4a). This is qualitatively similar to the behavior of the *survival* probability $P_S(t)$ [6,7,16] (but generally with different numerical value for the exponent). However, it is even qualitatively distinct from the behavior of the *return* probability

$$P_0(t) = C_{\mathbf{x}_0,\mathbf{x}_0}(t)/C(\mathbf{x}_0, t), \tag{3.7}$$

which was also calculated in Ref. [6] and found to be a power law in t for large t.

IV. DENSITY OF STATES

The transition probability matrix $\mathbf{W}$ is a random, non-negative definite matrix of size $S \times S$ where $S(\mathcal{C})$ is the number of available sites of a particular configuration $\mathcal{C}$ and, for our choice of the transition probabilities, it has elements $W_{\mathbf{x},\mathbf{y}}$ given by

$$W_{\mathbf{x},\mathbf{y}} = \begin{cases} 1/z & \text{if } |\mathbf{x}-\mathbf{y}| = 1 \text{ and } \mathbf{x},\mathbf{y} \in \mathcal{C} , \\ 0 & \text{otherwise} . \end{cases} \tag{4.1}$$

We will connect the spectral properties of this matrix to the behavior of the survival-return probability $P_S^0(t)$ in the same way as for the kinetic random walks [18]. This relation will be completely general and valid in any dimensions and any amount of disorder.

The uniformly weighted mean of $P_S^0(t)$ over all starting points of a given disorder configuration of size S is simply

$$\langle P_S^0(t)\rangle = \sum_{\mathbf{x}_0\in C} C_{\mathbf{x}_0,\mathbf{x}_0}(t)/(z^t S) \ . \tag{4.2}$$

For a given starting point, we have

$$C_{\mathbf{x}_0,\mathbf{x}_0}(t)/z^t = \mathbf{e}_{\mathbf{x}_0}^T \mathbf{W}^t \mathbf{e}_{\mathbf{x}_0} , \tag{4.3}$$

where $\mathbf{e}_{\mathbf{x}_0}$ is the column vector whose components are all zeros except the component (which is 1) corresponding to the site $\mathbf{x}_0$, and $\mathbf{e}_{\mathbf{x}_0}^T$ is the corresponding row vector of $\mathbf{e}_{\mathbf{x}_0}$. The superscript T refers to the transpose in the matrix notation. Thus,

$$\langle P_S^0(t)\rangle = \mathrm{Tr}\mathbf{W}^t/S = \sum_i {\lambda_i}^t/S \ , \tag{4.4}$$

where λ_i are the eigenvalues of $\mathbf{W}$.

For the case of a bipartite medium (e.g., a cluster on the square lattice or the simple cubic lattice), the eigenspectrum is symmetric about zero. In this case, $\langle P_S^0(t)\rangle$ is zero for all odd time steps t and twice the sum only over the positive eigenvalues for the even times t. (For non-bipartite cases this is no longer true, but the asymptotic long-time behavior should be largely determined by the part of the spectrum near the maximum in any case.) For the bipartite cases we have

$$\langle P_S^0(t)\rangle = (2/S)\sum_i e^{-\epsilon_i t} \ , \tag{4.5}$$

where the sum is over the positive spectrum and we let the eigenvalues $\lambda_i = e^{-\epsilon_i}$ with $\epsilon_i > 0$. Laplace transforming this, we get

$$\begin{aligned}\tilde{P}_S^0(\omega) = \mathcal{L}[\langle P_S^0(t)\rangle] &= (2/S)\sum_i \int_0^\infty dt e^{-(\omega+\epsilon_i)} \\ &= (2/S)\sum_i \frac{1}{\omega+\epsilon_i} \ .\end{aligned} \tag{4.6}$$

Then, using the well-known identity,

$$\frac{1}{x+i0^+} = P\left(\frac{1}{x}\right) - i\pi\delta(x) \ , \tag{4.7}$$

where P stands for the principal part in the sense of the distributions, we obtain

$$\rho(\epsilon) = -\frac{1}{\pi}\overline{\mathrm{Im}\tilde{P}_S^0(-\epsilon+i0^+)} \ , \tag{4.8}$$

where the disorder-averaged density of states $\rho(\epsilon)$ was defined to be

$$\rho(\epsilon) = \overline{(2/S)\sum_i \delta(\epsilon-\epsilon_i)}. \tag{4.9}$$

If we now assume a stretched-exponential behavior for the survival-return probability in the long-time limit as discussed earlier,

$$\ln \overline{P_S^0(t)} \sim -t^{\frac{\tilde{d}_0}{\tilde{d}_0+2}} \ , \tag{4.10}$$

then Eq. (4.8) predicts for $\epsilon \to 0$

$$\ln \rho(\epsilon) \sim -\epsilon^{-\tilde{d}_0/2} \ , \tag{4.11}$$

which is the corresponding Lifshitz tail for the density of states [17].

The exponent $\tilde{d}_0$, defined by Eq. (4.11), can be thought of as the analog of the spectral dimension [15] of the usual kinetic random walk. It should be stressed, however, that, unlike that case, $\tilde{d}_0$ is not a power law exponent and is not related to the *return* probability but to the *survival-return* probability.

V. NUMERICAL RESULTS

In order to numerically evaluate the density of states $\rho(\epsilon)$, we need to diagonalize a large number of large matrices since the thermodynamic limit of a large system is needed and a disorder average over many such systems must be taken. This would ordinarily present a challenge; however, a numerical method using the algorithm developed by Saad [19] conveniently allows us to diagonalize the matrices approximately and obtain the eigenspectrum near the extrema with high accuracy. This method was already successfully exploited in the context of the kinetic random walks (the so-called *ants*) [18].

Since the matrix $\mathbf{W}$ is symmetric, all the eigenvalues are real and, moreover, they are contained in the interval $(-1,1)$. The maximum eigenvalue λ_{max}, which has multiplicity 1 as ensured by the Perron-Frobenious theorem [20] for non-negative matrices, can be interpreted as the ratio z_{eff}/z of the effective coordination number to the full coordination number of the lattice. It should be noted, however, that $\mathbf{W}$ is *not* a Markov matrix, due to

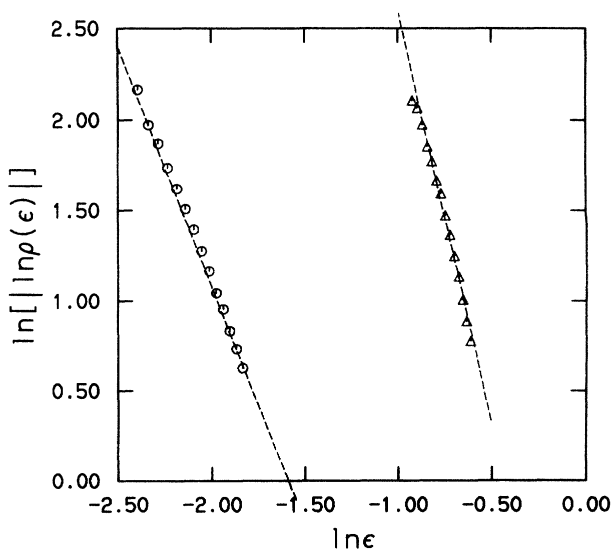

FIG. 3. Evaluation of the exponent $\tilde{d}_0/2$ for $d = 2$ (○) and $d = 3$ (△). The solid lines are the best fit results yielding $\tilde{d}_0/2 = 2.69\pm0.04$ and 4.42 ± 0.14 in $d = 2$ and 3, respectively. The corresponding χ_0 is given in Table II.

TABLE II. Estimates of the exponents $\tilde{d}_0$ and χ_0 from the density of states analysis. For comparison, those of χ_0 calculated from the survival-return probability are also repeated from Table I.

d	$\tilde{d}_0$	χ_0	χ_0 (D)[a]	χ_0 (I)[a]
2	5.38 ± 0.08	0.64 ± 0.01	0.65 ± 0.01	0.65 ± 0.01
3	8.84 ± 0.28	0.59 ± 0.01	0.72 ± 0.01	0.70 ± 0.01

[a] From Table I.

the nonconservation of the probability [21].

The disordered media were created as critical percolation clusters, starting from a seed site and growing layer by layer in a breadth-first fashion and stopping growth when a predetermined number of sites was generated. In both $d = 2$ and 3, we used clusters with a fixed size of 10 000, requiring **W** of $10\,000 \times 10\,000$. The exponent $\tilde{d}_0$ has been calculated from the numerically obtained density of states using Eq. (4.11) and the results are shown in Fig. 3. We employed 2500 configurations to extract the values of $\tilde{d}_0$ in both two and three dimensions. From Eqs. (3.6) and (4.10), the relation

$$\frac{\tilde{d}_0}{2} = \frac{2(1-\chi_0)}{2\chi_0 - 1} \tag{5.1}$$

follows, allowing the evaluation of χ_0 from that of $\tilde{d}_0$. The final exponent estimates from the density of states are summarized in Table II, where we have also repeated the estimates of χ_0 from Table I.

As shown in Fig. 3, the qualitative agreement with Eq. (4.11) is very clear for both $d = 2$ and 3. Quantitatively, the spectral and exact enumeration estimates of χ_0 seem to be in excellent agreement for $d = 2$, while those for $d = 3$ do not appear to agree as well. In particular, the trend with the dimensionality seems to be reversed depending on the method of evaluation. Also the value of z_{eff} from the maximum eigenvalue λ_{max} turns out to be 3.48 ± 0.04 and 3.67 ± 0.04 in $d = 2$ and $d = 3$, respectively. When compared with the exact enumeration estimates as discussed in Sec. III, it is apparent that the spectral calculation in $d = 2$ is much more consistent with the exact enumeration result than in $d = 3$.

The cause of this quantitative discrepancy is still unclear and it will require further work to determine whether it is simply a finite size effect or indicative of a deeper difference between two- and three-dimensional systems, perhaps due to the much less compact structure of the disordered media in three dimensions.

VI. LOCALIZATION PROPERTIES

It is well known that, in general, the localization properties of the eigenstates of a given Hamiltonian are strongly affected by the presence of quenched disorder [22]. Fewer efforts have been devoted, however, to the cases where the disorder is critical, as in the present problem.

We shall present here some preliminary results, based on the numerical investigations of the spectrum in the case of small matrices, but which nevertheless allow us to obtain a qualitative idea of the difference from the case in the absence of traps. A more detailed investigation along with a careful comparison with the case of the kinetic random walk are still in progress and will be published subsequently.

In Fig. 4 we show the entire spectra averaged over a number of typical configurations in $d = 2$ and 3. The symmetry about $\lambda = 0$ is due to the bipartite nature of the lattices used, and the absence of the eigenvalue

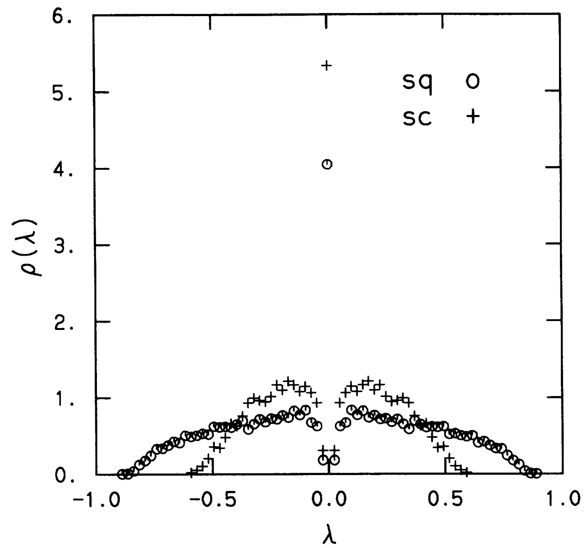

FIG. 4. Eigenspectra of a typical 400 × 400 transition probability matrix in $d = 2$ (○) and $d = 3$ (+) at critical disorder. "sq" refers to the square lattice and "sc" refers to simple cubic lattice.

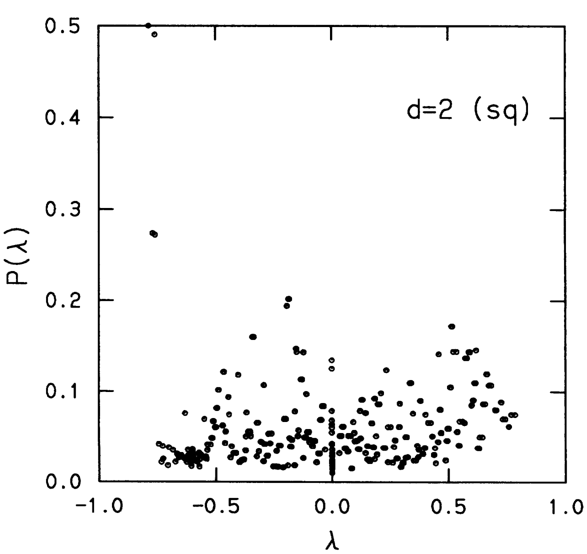

FIG. 5. Participation ratio $P(\lambda)$ for the spectrum of a 400 × 400 transition probability matrix in $d = 2$ at critical disorder. "sq" refers to the square lattice.

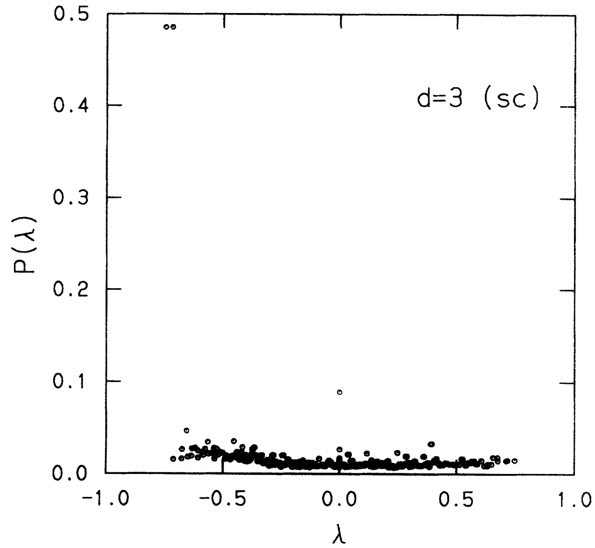

FIG. 6. Participation ratio $P(\lambda)$ for the spectrum of a 400×400 transition probability matrix in $d = 3$ at critical disorder. "sc" refers to the simple cubic lattice.

with magnitude 1 (or close to it) reflects the lack of the conservation of probability. Clearly there is no trace of the accumulation of modes near the maximum as is the case for the ants [18].

A quantitative measurement of the degree of localization of a mode is given by the *participation ratio* [23]:

$$P(\lambda) = \frac{\sum_{\mathbf{x}} |u_{\mathbf{x}}(\lambda)|^4}{(\sum_{\mathbf{x}} |u_{\mathbf{x}}(\lambda)|^2)^2} , \tag{6.1}$$

where $u_{\mathbf{x}}$ is the amplitude of the mode at $\mathbf{x}$. With this definition, we have $P(\lambda) \in [0,1]$ and a value close to 1 corresponds to a well localized state, while a small value (of order of $1/S$ where S is the number of sites in the cluster $\mathcal{C}$) corresponds to an extended state.

We diagonalized exactly a small matrix of 400×400 for both $d = 2$ and $d = 3$. The results for $P(\lambda)$ are presented in Figs. 5 and 6 in $d = 2$ and 3, respectively. Even though all states are localized, it appears that some modes are particularly strongly localized, such as those corresponding to the extreme negative part of the spectrum, while some more closely resemble an extended state. To illustrate this, we plot in Fig. 7 the amplitudes of some chosen modes in the $d = 2$ case. Figures 7(a) and 7(b) correspond to the two extremal eigenvalues (negative and positive) and 7(c) to a less localized state at $\lambda = 0$. These structures seem to be consistent with their participation ratios. Similar results persist also for different configurations.

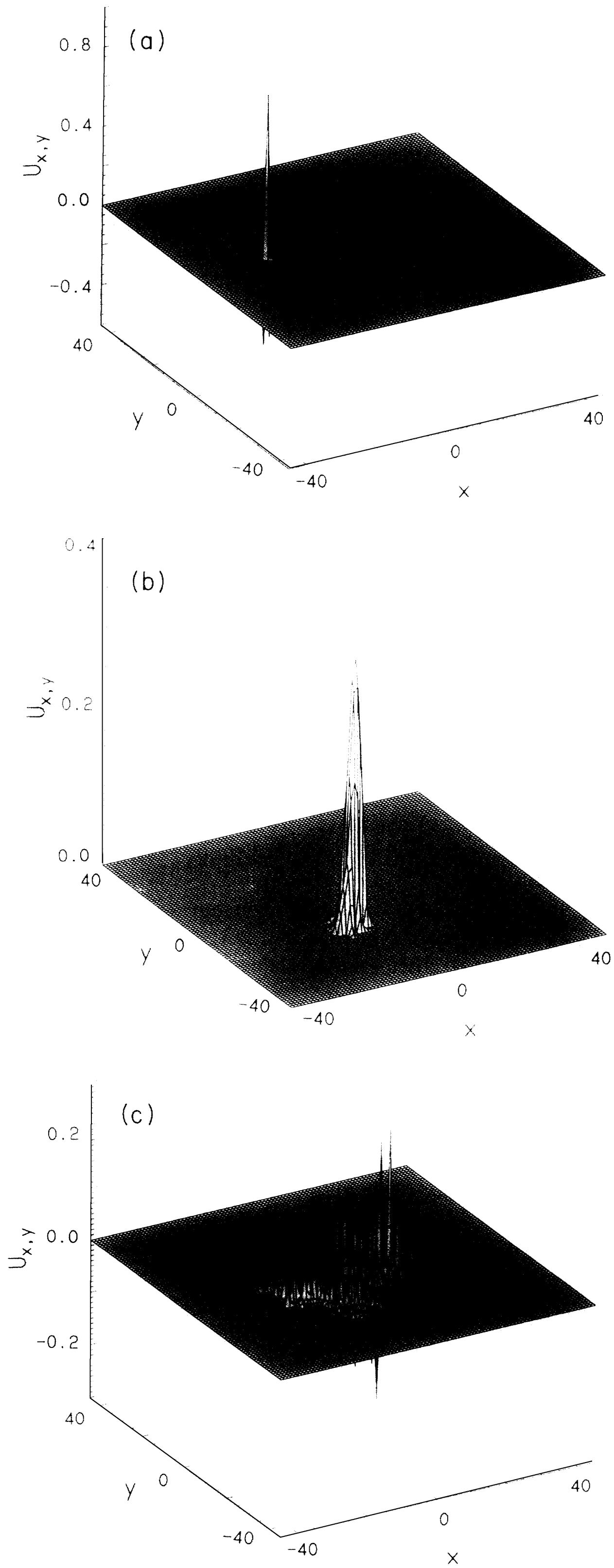

FIG. 7. Plot of the $\mathbf{x} = (x, y)$ dependence of the amplitude $u_{\mathbf{x}}(\lambda)$ of the eigenmodes corresponding to the lower ($\lambda = -0.8390\ldots$), upper ($\lambda = 0.8390\ldots$), and middle ($\lambda = 0$) part of the spectrum [in (a), (b), and (c), respectively] for the two-dimensional case.

VII. CONCLUSIONS

The main aim of the present work is to confirm the existence of a stretched-exponential tail in the long-time behavior of the various probabilities for diffusion with critically disordered traps. This has been accomplished by comparing the predicted values of the appropriate exponents based on the exact enumeration results with the corresponding Lifshitz tail in the density of states of the transition probability matrix.

Numerical results indicate that the two procedures give *qualitatively* consistent results in both two and three dimensions. *Quantitative* agreement is also excellent in two dimensions; however, it is substantially poorer in three dimensions. More work will be necessary to determine the cause of this numerical discrepancy.

The localization properties of the eigenstates of this system were also investigated using mainly the participation ratio. It was found that, although all states seem to have a localized character, a wide range of the degree of localization, ranging from almost extended to fully localized, is present. In particular, the lowest extreme of the spectrum (in the negative part, which does not seem to play any role in the determination of the long-time diffusional behavior) is extremely localized.

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[16] As pointed out in [6], the analysis of the power moments of C given in Ref. [5] appears to be incorrect, having been based on insufficient numerical evidence.

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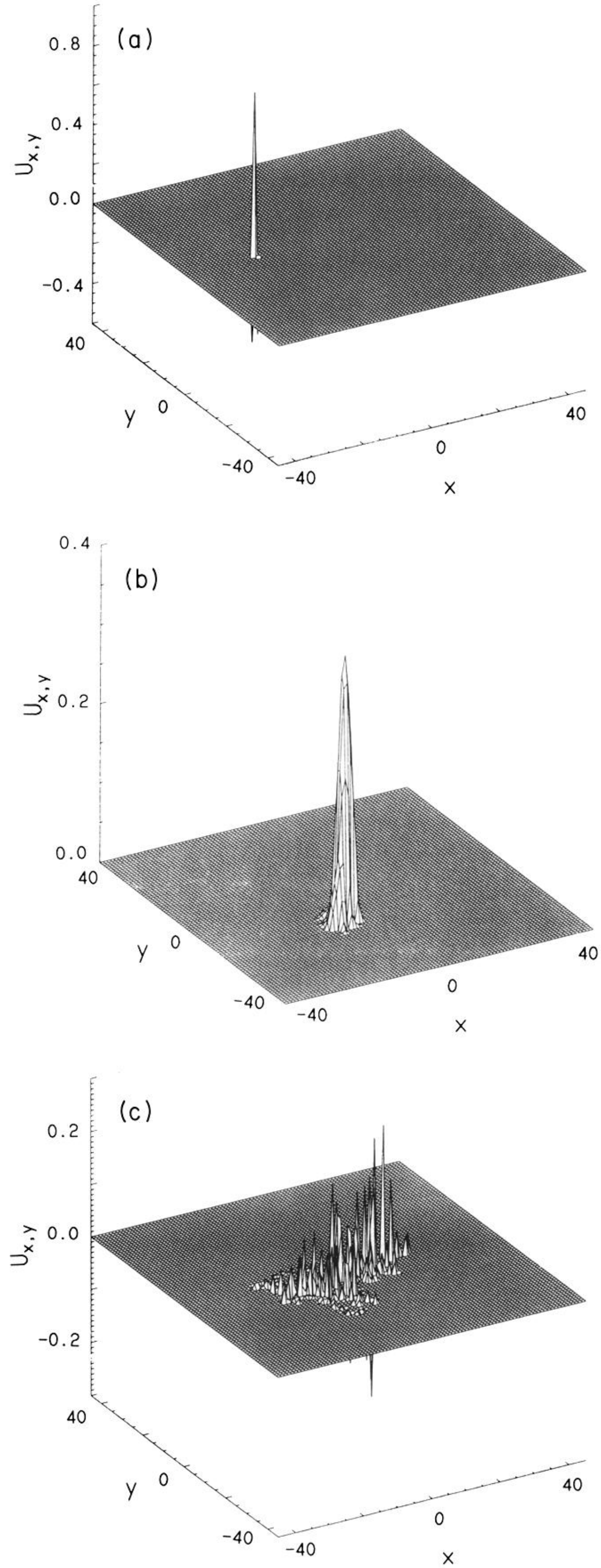

FIG. 7. Plot of the $\mathbf{x} = (x, y)$ dependence of the amplitude $u_{\mathbf{x}}(\lambda)$ of the eigenmodes corresponding to the lower ($\lambda = -0.8390\ldots$), upper ($\lambda = 0.8390\ldots$), and middle ($\lambda = 0$) part of the spectrum [in (a), (b), and (c), respectively] for the two-dimensional case.